Problem 1

Take the particle to be at rest in the system K', moving along the x axis at velocity v with respect to the system K.

$$\vec{E'} = \frac{q\vec{r'}}{r'^3}, \qquad \vec{B'} = 0 \tag{1}$$

Now, let us analyse the fields in the system K, in the plane $z = 0$ (there is rotational symmetry along the x axis).

$$E_x = E'_x, \qquad E_y = E'_y \gamma \tag{2}$$

$$\vec{E'} = \frac{q}{r'^3}(x'\hat{x} + y'\hat{y}) \tag{3}$$

$$\vec{E} = \frac{q}{r'^3}(x'\hat{x} + y'\gamma\hat{y}) \tag{4}$$

$$\vec{E} = \frac{q\gamma}{r'^3}(x'\gamma^{-1}\hat{x} + y'\hat{y}) \tag{5}$$

But, given the position of a point, measured at any time in the system K' (the charge is at rest in that frame), the value of x and y at the time t will be determined by

$$x' = (x - vt)\gamma, \qquad y' = y \tag{6}$$

$$\vec{E} = q\frac{\gamma}{((x-vt)^2\gamma^2 + y^2 + z^2)^{3/2}}\left((x - vt)\hat{x} + y\hat{y} + z\hat{z}\right) \tag{7}$$

where we have used the symmetry under rotations to generalize the value of the field in the z-direction.

Therefore, the field is radial with respect to the particle at any time t, but not spherically symmetric, due to the appearence of the γ^2 factor along the x-direction.

What about the magnetic field?

$$B_x = B'_x = 0, \qquad B_z = \frac{v}{c}E'_y\gamma \tag{8}$$

The field is perpendicular to the direction of motion (and symmetric under rotations along the x axis). Since $E_y = E'_y\gamma$, then

$$|\vec{B}| = qv/c\frac{\gamma(y^2+z^2)^{1/2}}{((x-vt)^2\gamma^2+y^2+z^2)^{3/2}} \tag{9}$$

Observe that, as the velocity approaches the speed of light (take $t=0$, for simplicity), then the electric field is still radial, but

$$\vec{E} = q\frac{\vec{r}}{\gamma^2(x^2+y^2\gamma^{-2})^{3/2}} \tag{10}$$

The magnitude along the x-axis at a fixed time in the reference frame K tends to zero, at any non-vanishing separation r from the moving charge.

A test particle at rest in the axis of motion in the system K at time $t=0$ will feel a force acting on it which vanishes as the moving particle velocity approaches c, even if it is very close to the original charged particle !
What is the description of this process in the system K'?

The separation between the moving particle and the test particle in K' is given by:

$$x' = x\gamma \tag{11}$$

Therefore, as the velocity approaches c, the observer in K' will see that the test particle and the moving particles are separated by asymptotically large distances, independently of $x \neq 0$. The force vanishes also in K', due to this. Indeed since the magnetic field vanishes along the x-axis, the force is purely electric, and $E_x(x,0) = E'_x(x',0)$.

Problem 2.

The easiest way of solving this problem is to define the plane defined by the magnetic and electric fields as the yz plane, and move to a reference frame K' which moves at a constant speed u_x with respect to the original frame K. The requirement of parallelism of the electric and magnetic fields is equivalent to the condition that $E\text{x}B = 0$. Or, equivalently,

$$\frac{E'_z}{E'_y} = \frac{B'_z}{B'_y} \tag{12}$$

Using the transformation laws

$$\begin{aligned} E'_y &= \gamma\left(E_y - \frac{u_x}{c}B_z\right) \qquad\qquad B'_y = \gamma\left(B_y + \frac{u_x}{c}E_z\right) \\ E'_z &= \gamma\left(E_z + \frac{u_x}{c}B_y\right) \qquad\qquad B'_z = \gamma\left(B_z - \frac{u_x}{c}E_y\right) \end{aligned} \tag{13}$$

and Eq. (12), we get the relation

$$(E_yB_z - E_zB_y) - \frac{u_x}{c}\left(\vec{E}^2 + \vec{B}^2\right) + \frac{u_x^2}{c^2}(E_yB_z - E_zB_y) = 0. \tag{14}$$

The velocity u_x then fulfills the relation

$$\frac{u_x/c}{u_x^2/c^2 + 1} = \frac{E_yB_z - E_zB_y}{\vec{E}^2 + \vec{B}^2}. \tag{15}$$

In vectorial form

$$\frac{\vec{u}/c}{\vec{u}^2/c^2 + 1} = \frac{E\mathrm{x}B}{\vec{E}^2 + \vec{B}^2}. \tag{16}$$

The solution for u_x is

$$\frac{|u_x|}{c} = \frac{|\alpha|}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - 1} \tag{17}$$

with

$$\alpha = \frac{\vec{E}^2 + \vec{B}^2}{E_yB_z - B_yE_z} \tag{18}$$

Three comments:
1) For fields which are almost parallel to each other, the solution is simply $|u_x/c| \simeq 1/|\alpha|$, which is indeed a small number.

2) The above equation seems to have solutions even in the case in which the fields are perpendicular to each other. However, we know that in this case it is impossible to find a system of reference in which they are parallel to each other. The solution to this paradox is that the obtained velocity u_x in Eq. (17) is the appropriate one to make either the magnetic or electric field vanish. Observe also that if the fields are perpendicular and of equal magnitude, we obtain $|\alpha| = 2$, or $|u_x| = c$. Therefore, there is no physical

inertial reference system leading to the desired solution. (With physical I mean one in which I can put an observer and a system of clocks).

3) Once I found a solution to the above problem, I can find infinitely many more by choosing other reference frames moving at constant speed in the direction of the fields in the system K'.

Problem 3.

Let us take the magnetic and electric fields pointing in the direction of the x-axis, $\vec{E} = E\hat{x}$, $\vec{B} = B\hat{x}$.
Then, the equations of motion are

$$\frac{d}{dt}\left(\frac{\mathcal{E}u_x}{c^2}\right) = \frac{d}{dt}p_x = qE \tag{19}$$

$$\frac{dp_y}{dt} = -qB\frac{u_z}{c} \tag{20}$$

$$\frac{dp_z}{dt} = qB\frac{u_y}{c} \tag{21}$$

where $\mathcal{E}$ is the energy of the particle.
From the first equation, we obtain,

$$u_x = \frac{c^2 qEt}{\mathcal{E}} \tag{22}$$

The Energy variation is given by

$$\frac{d}{dt}\mathcal{E} = qEu_x = \frac{c^2 q^2 E^2 t}{\mathcal{E}} \tag{23}$$

Therefore,

$$\mathcal{E} = \sqrt{\mathcal{E}_0^2 + c^2 q^2 E^2 t^2} \tag{24}$$

But this may be rewritten in terms of $p_x = \mathcal{E}u_x/c^2$. Indeed,

$$\mathcal{E} = \sqrt{\mathcal{E}_0^2 + c^2 p_x^2} \tag{25}$$

From here, remembering that, in general,

$$\mathcal{E}^2 = m^2c^4 + c^2p_y^2 + c^2p_z^2 + c^2p_x^2 \tag{26}$$

we get that the modulus of the momentum in the direction perpendicular to the fields remains constant. In addition, they should fulfill the equation

$$\frac{d}{dt}\left(p_y + ip_z\right) = i\frac{qc\,B}{\mathcal{E}}(p_y + ip_z) \tag{27}$$

But, since we know that the modulus should remain constant, the solution to this equation should be of the form

$$p_y + ip_z = p_0 \exp(i(\phi + \alpha)) \tag{28}$$

with α an arbitrary constant and

$$\frac{d\phi}{dt} = \frac{qcB}{\mathcal{E}} \tag{29}$$

or, equivalently

$$\phi = \frac{B}{E}\sinh^{-1}\left(\frac{qEct}{\mathcal{E}_0}\right) \tag{30}$$

Observe that, for $E \to 0$, this leads to a constant frequency equal to $\omega = qcB/\mathcal{E}_0$.

For $\alpha = 0$, we get $p_y = p_0\cos(\phi)$, $p_z = p_0 sin(\phi)$, meaning that the particle at time $t = 0$ has no momentum in the z direction. It is now trivial to obtain the trajectory. Let us assume that the particle is at the origin at the time $t = 0$. Taking the expression of u_x, Eq. (22), we obtain, by integration,

$$x = \frac{1}{qE}\left(\sqrt{\mathcal{E}_0^2 + c^2q^2E^2t^2} - \mathcal{E}_0\right) \tag{31}$$

while the equations for the y and z corrdinates can be easily integrated out, leading to

$$y = \frac{cp_0}{qB}\sin\phi \tag{32}$$

$$z = \frac{cp_0}{qB}\left(-\cos\phi + 1\right). \tag{33}$$

The particle moves along an helix, with fixed radius but increasing step. The angular velocity decreases as the energy increases.

We already commented that for $E \to 0$, we get a constant frequency. In addition, both the energy and the momentum and velocity along the x-direction become constant. Therefore, we recover the equations of a particle moving in a constant magnetic field. The trajectory is an helix with fixed radius and step.

For $B \to 0$, instead, ϕ tends to zero linearly with B. Hence, replacing $\sin\phi$ with ϕ and $\cos\phi$ by 1, we get that, while the equation of motion in the x direction is not modified,

$$y = \frac{cp_0}{qE} \sinh^{-1}\left(\frac{qEct}{\mathcal{E}_0}\right) \tag{34}$$

and $z = 0$. Therefore, we recover the solution for a particle in a constant electric field.